

NOTES ON THE GABRIEL-ROITER MEASURE

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In his proof of the first Brauer-Thrall conjecture [6], Roiter used an induction scheme which Gabriel formalized in his report on abelian length categories [2]. The first Brauer-Thrall conjecture asserts that every finite dimensional algebra of bounded representation type is of finite representation type. Ringel noticed¹ that the formalism of Gabriel and Roiter is also useful for studying the representations of algebras having unbounded representation type.

In these notes we present a purely combinatorial definition of the Gabriel-Roiter measure and combine this with an axiomatic characterization; see also [3]. Given a finite dimensional algebra Λ , the Gabriel-Roiter measure is characterized as a universal morphism $\text{ind } \Lambda \rightarrow P$ of partially ordered sets. The map is defined on the isomorphism classes of finite dimensional indecomposable Λ -modules and is a suitable refinement of the length function $\text{ind } \Lambda \rightarrow \mathbb{N}$ which sends a module to its composition length. The axiomatic treatment is complemented by a recursive definition of the Gabriel-Roiter measure.

The second part of these notes discusses the Gabriel-Roiter measure for a fixed abelian length category. This is the original setting for Gabriel's work. In particular, Gabriel's main property of the measure is proved. This is used to extend the Gabriel-Roiter measure from indecomposable to arbitrary objects. Our main example is the category of finite dimensional Λ -modules over some finite dimensional algebra Λ . We report on Ringel's work [4, 5], presenting for instance his refinement of the first Brauer-Thrall conjecture.

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1. CHAINS AND LENGTH FUNCTIONS

1.1. The Gabriel-Roiter measure. There are a number of possible approaches to define the Gabriel-Roiter measure. Fix a partially ordered set (S, \leq) which is equipped with a length function $\lambda: S \rightarrow \mathbb{N}$. We start off by defining the Gabriel-Roiter measure for S as a morphism $\mu: S \rightarrow P$ of partially ordered sets which refines the length function λ . Let us stress right away that the values $\mu(x)$ for $x \in S$ are not relevant. All we need to know is whether for a pair x, y of elements in S , the relation $\mu(x) \leq \mu(y)$ holds or not. This is the essence of a measure and we make this precise in the following definition.

¹Cf. the footnote on p. 91 of [2].

Definition. Let (S, \leq) be a partially ordered set. A *measure* μ for S is a relation on S , written $\mu(x) \leq \mu(y)$, for a pair x, y of elements in S , such that for all x, y, z in S the following holds:

- (M1) $\mu(x) \leq \mu(y)$ and $\mu(y) \leq \mu(z)$ imply $\mu(x) \leq \mu(z)$.
- (M2) $\mu(x) \leq \mu(y)$ or $\mu(y) \leq \mu(x)$.
- (M3) $x \leq y$ implies $\mu(x) \leq \mu(y)$.

We write $\mu(x) = \mu(y)$ if both $\mu(x) \leq \mu(y)$ and $\mu(y) \leq \mu(x)$ hold.

A measure μ for S gives rise to an equivalence relation on S as follows: Call two elements x and y *equivalent* if $\mu(x) = \mu(y)$. The set S/μ of equivalence classes is totally ordered via μ and the canonical map $S \rightarrow S/\mu$ is a morphism of partially ordered sets. Conversely, any morphism $\phi: S \rightarrow P$ to a totally ordered set P gives rise to a measure μ for S provided one defines $\mu(x) \leq \mu(y)$ if $\phi(x) \leq \phi(y)$ holds.

In this section we present three different approaches defining the Gabriel-Roiter measure for a partially ordered set S and a length function $\lambda: S \rightarrow \mathbb{N}$. To be more precise, we define the Gabriel-Roiter measure as a morphism $S \rightarrow \text{Ch}(\mathbb{N})$ of partially ordered sets, where $\text{Ch}(\mathbb{N})$ denotes the lexicographically ordered set of finite sets of natural numbers. We complement this by a recursive and an axiomatic definition. Note that all three concepts are equivalent in the sense that they yield the same measure for S .

1.2. The lexicographic order on finite chains. Let (S, \leq) be a partially ordered set. A subset $X \subseteq S$ is a *chain* if $x_1 \leq x_2$ or $x_2 \leq x_1$ for each pair $x_1, x_2 \in X$. For a finite chain X , we denote by $\min X$ its minimal and by $\max X$ its maximal element, using the convention

$$\max \emptyset < x < \min \emptyset \quad \text{for all } x \in S.$$

We write $\text{Ch}(S)$ for the set of all finite chains in S and let

$$\text{Ch}(S, x) := \{X \in \text{Ch}(S) \mid \max X = x\} \quad \text{for } x \in S.$$

On $\text{Ch}(S)$ we consider the *lexicographic order* which is defined by

$$X \leq Y \iff \min(Y \setminus X) \leq \min(X \setminus Y) \quad \text{for } X, Y \in \text{Ch}(S).$$

Remark. (1) $X \subseteq Y$ implies $X \leq Y$ for $X, Y \in \text{Ch}(S)$.

(2) Suppose that S is totally ordered. Then $\text{Ch}(S)$ is totally ordered. We may think of $X \in \text{Ch}(S) \subseteq \{0, 1\}^S$ as a string of 0s and 1s which is indexed by the elements in S . The usual lexicographic order on such strings coincides with the lexicographic order on $\text{Ch}(S)$.

Example. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and \mathbb{Q} be the set of rational numbers together with the natural ordering. Then the map

$$\text{Ch}(\mathbb{N}) \longrightarrow \mathbb{Q}, \quad X \mapsto \sum_{x \in X} 2^{-x}$$

is injective and order preserving, taking values in the interval $[0, 1]$. For instance, the subsets of $\{1, 2, 3\}$ are ordered as follows:

$$\{\} < \{3\} < \{2\} < \{2, 3\} < \{1\} < \{1, 3\} < \{1, 2\} < \{1, 2, 3\}.$$

We need the following properties of the lexicographic order.

Lemma. Let $X, Y \in \text{Ch}(S)$ and $X^* := X \setminus \{\max X\}$.

- (1) $X^* = \max\{X' \in \text{Ch}(S) \mid X' < X \text{ and } \max X' < \max X\}$.
- (2) If $X^* < Y$ and $\max X \geq \max Y$, then $X \leq Y$.

Proof. (1) Let $X' < X$ and $\max X' < \max X$. We show that $X' \leq X^*$. This is clear if $X' \subseteq X^*$. Otherwise, we have

$$\min(X^* \setminus X') = \min(X \setminus X') < \min(X' \setminus X) = \min(X' \setminus X^*),$$

and therefore $X' \leq X^*$.

- (2) The assumption $X^* < Y$ implies by definition

$$\min(Y \setminus X^*) < \min(X^* \setminus Y).$$

We consider two cases. Suppose first that $X^* \subseteq Y$. If $X \subseteq Y$, then $X \leq Y$. Otherwise,

$$\min(Y \setminus X) < \max X = \min(X \setminus Y)$$

and therefore $X < Y$. Now suppose that $X^* \not\subseteq Y$. We use again that $\max X \geq \max Y$, exclude the case $Y \subseteq X$, and obtain

$$\min(Y \setminus X) = \min(Y \setminus X^*) < \min(X^* \setminus Y) = \min(X \setminus Y).$$

Thus $X \leq Y$ and the proof is complete. \square

1.3. Length functions. Let (S, \leq) be a partially ordered set. A *length function* on S is by definition a map

$$\lambda: S \longrightarrow \mathbb{N} = \{1, 2, 3, \dots\}$$

such that $x < y$ in S implies $\lambda(x) < \lambda(y)$. A length function $\lambda: S \rightarrow \mathbb{N}$ induces for each $x \in S$ a map

$$\text{Ch}(S, x) \longrightarrow \text{Ch}(\mathbb{N}, \lambda(x)), \quad X \mapsto \lambda(X),$$

and therefore the following *chain length function*

$$S \longrightarrow \text{Ch}(\mathbb{N}), \quad x \mapsto \lambda^*(x) := \max\{\lambda(X) \mid X \in \text{Ch}(S, x)\}.$$

This chain length function is by definition the *Gabriel-Roiter measure* for S with respect to λ .

We continue with a list of basic properties (C0) – (C5) of λ^* .

1.4. A recursive definition. The following property (C0) of the chain length function $\lambda^*: S \rightarrow \text{Ch}(\mathbb{N})$ can be used to define λ^* by induction on the length of the elements in S . We take this as our second definition of the Gabriel-Roiter measure for S with respect to λ . Note that $\lambda^*(x) = \{\lambda(x)\}$ if x is a minimal element of S .

Proposition. Let $x \in S$.

$$(C0) \quad \lambda^*(x) = \max_{x' < x} \lambda^*(x') \cup \{\lambda(x)\}.$$

Proof. Let $X = \lambda^*(x)$ and note that $\max X = \lambda(x)$. The assertion follows from Lemma 1.2 because we have

$$X \setminus \{\max X\} = \max\{X' \in \text{Ch}(\mathbb{N}) \mid X' < X \text{ and } \max X' < \max X\}. \quad \square$$

1.5. Basic properties. Let $\lambda: S \rightarrow \mathbb{N}$ be a length function and $\lambda^*: S \rightarrow \text{Ch}(\mathbb{N})$ the induced chain length function. The following basic properties suggest to think of λ^* as a refinement of λ .

Proposition. *Let $x, y \in S$.*

- (C1) $x \leq y$ implies $\lambda^*(x) \leq \lambda^*(y)$.
- (C2) $\lambda^*(x) = \lambda^*(y)$ implies $\lambda(x) = \lambda(y)$.
- (C3) $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ and $\lambda(x) \geq \lambda(y)$ imply $\lambda^*(x) \leq \lambda^*(y)$.

Proof. Suppose $x \leq y$ and let $X \in \text{Ch}(S, x)$. Then $Y = X \cup \{y\} \in \text{Ch}(S, y)$ and we have $\lambda(X) \leq \lambda(Y)$ since $\lambda(X) \subseteq \lambda(Y)$. Thus $\lambda^*(x) \leq \lambda^*(y)$. If $\lambda^*(x) = \lambda^*(y)$, then

$$\lambda(x) = \max \lambda^*(x) = \max \lambda^*(y) = \lambda(y).$$

To prove (C3), we use (C0) and apply Lemma 1.2 with $X = \lambda^*(x)$ and $Y = \lambda^*(y)$. In fact, $\lambda^*(x') < \lambda^*(y)$ for all $x' < x$ implies $X^* < Y$, and $\lambda(x) \geq \lambda(y)$ implies $\max X \geq \max Y$. Thus $X \leq Y$. \square

We state some further elementary properties of the map λ^* .

Proposition. *Let $x, y \in S$.*

- (C4) $\lambda^*(x) \leq \lambda^*(y)$ or $\lambda^*(y) \leq \lambda^*(x)$.
- (C5) $\{\lambda^*(x) \mid x \in S \text{ and } \lambda(x) \leq n\}$ is finite for all $n \in \mathbb{N}$.

Proof. (C4) is clear since $\text{Ch}(\mathbb{N})$ is totally ordered. (C5) follows from the fact that $\{X \in \text{Ch}(\mathbb{N}) \mid \max X \leq n\}$ is finite for all $n \in \mathbb{N}$. \square

The map λ^* induces a measure μ for S in the sense of Definition 1.1.

Corollary. *The chain length function λ^* induces via*

$$\mu(x) \leq \mu(y) \quad :\Longleftrightarrow \quad \lambda^*(x) \leq \lambda^*(y) \quad \text{for } x, y \in S$$

a measure for S . Moreover, we have for all x, y in S

$$\mu(x) = \mu(y) \quad \Longleftrightarrow \quad \max_{x' < x} \mu(x') = \max_{y' < y} \mu(y') \text{ and } \lambda(x) = \lambda(y).$$

Proof. (C1) and (C4) imply that the map λ^* induces a measure μ for S . The characterization for $\mu(x) = \mu(y)$ follows from (C0). \square

1.6. An axiomatic definition. Let $\lambda: S \rightarrow \mathbb{N}$ be a length function. We present an axiomatic characterization of the induced chain length function λ^* . Thus we can replace the original definition in terms of chains by three simple conditions which express the fact that λ^* refines λ . We take this as our third definition of the Gabriel-Roiter measure for S with respect to λ .

Theorem. *Let $\lambda: S \rightarrow \mathbb{N}$ be a length function. Then there exists a map $\mu: S \rightarrow P$ into a partially ordered set P satisfying for all $x, y \in S$ the following:*

- (P1) $x \leq y$ implies $\mu(x) \leq \mu(y)$.
- (P2) $\mu(x) = \mu(y)$ implies $\lambda(x) = \lambda(y)$.
- (P3) $\mu(x') < \mu(y)$ for all $x' < x$ and $\lambda(x) \geq \lambda(y)$ imply $\mu(x) \leq \mu(y)$.

Moreover, for any map $\mu': S \rightarrow P'$ into a partially ordered set P' satisfying the above conditions, we have for all x, y in S

$$\mu'(x) \leq \mu'(y) \quad \Longleftrightarrow \quad \mu(x) \leq \mu(y) \quad \Longleftrightarrow \quad \lambda^*(x) \leq \lambda^*(y).$$

Proof. We have seen in (1.5) that λ^* satisfies (P1) – (P3). So it remains to show that for any map $\mu: S \rightarrow P$ into a partially ordered set P , the conditions (P1) – (P3) uniquely determine the relation $\mu(x) \leq \mu(y)$ for any pair $x, y \in S$. In fact, we claim that (P1) – (P3) imply $\mu(x) \leq \mu(y)$ or $\mu(y) \leq \mu(x)$. We proceed by induction on the length of the elements in S . For elements of length $n = 1$, the assertion is clear. In fact, $\lambda(x) = 1 = \lambda(y)$ implies $\mu(x) = \mu(y)$ by (P3). Now let $n > 1$ and assume the assertion is true for all elements $x \in S$ of length $\lambda(x) < n$. We choose for each $x \in S$ of length $\lambda(x) \leq n$ a *Gabriel-Roiter filtration*, that is, a sequence

$$x_1 < x_2 < \dots < x_{\gamma(x)-1} < x_{\gamma(x)} = x$$

in S such that x_1 is minimal and $\max_{x' < x_i} \mu(x') = \mu(x_{i-1})$ for all $1 < i \leq \gamma(x)$. Such a filtration exists because the elements $\mu(x')$ with $x' < x$ are totally ordered. Now fix $x, y \in S$ of length at most n and let $I = \{i \geq 1 \mid \mu(x_i) = \mu(y_i)\}$. We consider $r = \max I$ and put $r = 0$ if $I = \emptyset$. There are two possible cases. Suppose first that $r = \gamma(x)$ or $r = \gamma(y)$. If $r = \gamma(x)$, then $\mu(x) = \mu(x_r) = \mu(y_r) \leq \mu(y)$ by (P1). Now suppose $\gamma(x) \neq r \neq \gamma(y)$. Then we have $\lambda(x_{r+1}) \neq \lambda(y_{r+1})$ by (P2) and (P3). If $\lambda(x_{r+1}) > \lambda(y_{r+1})$, then we obtain $\mu(x_{r+1}) < \mu(y_{r+1})$, again using (P2) and (P3). Iterating this argument, we get $\mu(x) = \mu(x_{\gamma(x)}) < \mu(y_{r+1})$. From (P1) we get $\mu(x) < \mu(y_{r+1}) \leq \mu(y)$. Thus $\mu(x) \leq \mu(y)$ or $\mu(y) \leq \mu(x)$ and the proof is complete. \square

2. ABELIAN LENGTH CATEGORIES

2.1. Additive categories. A category \mathcal{A} is *additive* if every finite family X_1, X_2, \dots, X_n of objects has a coproduct

$$X_1 \oplus X_2 \oplus \dots \oplus X_n,$$

each set $\text{Hom}_{\mathcal{A}}(A, B)$ is an abelian group, and the composition maps

$$\text{Hom}_{\mathcal{A}}(B, C) \times \text{Hom}_{\mathcal{A}}(A, B) \longrightarrow \text{Hom}_{\mathcal{A}}(A, C)$$

are bilinear.

2.2. Abelian categories. An additive category \mathcal{A} is *abelian*, if every map $\phi: A \rightarrow B$ has a kernel and a cokernel, and if the canonical factorization

$$\begin{array}{ccccccc} \text{Ker } \phi & \xrightarrow{\phi'} & A & \xrightarrow{\phi} & B & \xrightarrow{\phi''} & \text{Coker } \phi \\ & & \downarrow & & \uparrow & & \\ & & \text{Coker } \phi' & \xrightarrow{\bar{\phi}} & \text{Ker } \phi'' & & \end{array}$$

of ϕ induces an isomorphism $\bar{\phi}$.

Example. The category of modules over any associative ring is an abelian category.

2.3. Subobjects. Let \mathcal{A} be an abelian category. We say that two monomorphisms $X_1 \rightarrow X$ and $X_2 \rightarrow X$ are *equivalent*, if there exists an isomorphism $X_1 \rightarrow X_2$ making the following diagram commutative.

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad} & X_2 \\ & \searrow & \swarrow \\ & X & \end{array}$$

An equivalence class of monomorphisms into X is called a *subobject* of X . Given subobjects $X_1 \rightarrow X$ and $X_2 \rightarrow X$, we write $X_1 \subseteq X_2$ if there is a morphism $X_1 \rightarrow X_2$ making

the above diagram commutative. An object $X \neq 0$ is *simple* if $X' \subseteq X$ implies $X' = 0$ or $X' = X$.

2.4. Length categories. Let \mathcal{A} be an abelian category. An object X has *finite length* if it has a finite composition series

$$0 = X_0 \subseteq X_1 \subseteq \dots \subseteq X_{n-1} \subseteq X_n = X,$$

that is, each X_i/X_{i-1} is simple. In this case the length of a composition series is an invariant of X by the Jordan-Hölder Theorem; it is called the *length* of X and is denoted by $\ell(X)$. For instance, X is simple if and only if $\ell(X) = 1$. Note that X has finite length if and only if X is both artinian (i.e. satisfies the descending chain condition on subobjects) and noetherian (i.e. satisfies the ascending chain condition on subobjects).

An abelian category is called a *length category* if all objects have finite length and the isomorphism classes of objects form a set.

An object $X \neq 0$ is called *indecomposable* if $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$. A finite length object admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism by the Krull-Remak-Schmidt Theorem.

We denote by $\text{ind } \mathcal{A}$ the set of isomorphism classes of indecomposable objects of \mathcal{A} .

Example. (1) Let Λ be an artinian ring. Then the category of finitely generated Λ -modules form a length category which we denote by $\text{mod } \Lambda$.

(2) Let k be a field and Q be any quiver. Then the finite dimensional k -linear representations of Q form a length category.

3. THE GABRIEL-ROITER MEASURE

Let \mathcal{A} be an abelian length category. We give the definition of the Gabriel-Roiter measure for \mathcal{A} which is due to Gabriel [2] and was inspired by the work of Roiter [6]. Then we discuss some specific properties, including Ringel's results about Gabriel-Roiter inclusions [4].

3.1. The definition. Let \mathcal{A} be an abelian length category. The isomorphism classes of objects of \mathcal{A} are partially ordered via the subobject relation

$$X \subseteq Y \quad :\Longleftrightarrow \quad \text{there exists a monomorphism } X \rightarrow Y.$$

We consider the length function $\ell: \text{ind } \mathcal{A} \rightarrow \mathbb{N}$ which takes an object X to its composition length $\ell(X)$. Then the induced chain length function $\ell^*: \text{ind } \mathcal{A} \rightarrow \text{Ch}(\mathbb{N})$ is by definition the *Gabriel-Roiter measure* for \mathcal{A} . We will only work with this definition when making explicit computations. Otherwise, we take the induced measure in the sense of Definition 1.1 which is characterized as follows.

Theorem. *Let \mathcal{A} be an abelian length category. The Gabriel-Roiter measure induces via*

$$\mu(X) \leq \mu(Y) \quad :\Longleftrightarrow \quad \ell^*(X) \leq \ell^*(Y) \quad \text{for } X, Y \in \text{ind } \mathcal{A}$$

a relation on $\text{ind } \mathcal{A}$. This is the unique transitive relation on $\text{ind } \mathcal{A}$ satisfying for all objects X, Y the following:

- (GR1) $X \subseteq Y$ implies $\mu(X) \leq \mu(Y)$.
- (GR2) $\mu(X) = \mu(Y)$ implies $\ell(X) = \ell(Y)$.
- (GR3) $\mu(X') < \mu(Y)$ for all $X' \subset X$ and $\ell(X) \geq \ell(Y)$ imply $\mu(X) \leq \mu(Y)$.

Here we use the following convention: We write $\mu(X) = \mu(Y)$ if $\mu(X) \leq \mu(Y)$ and $\mu(Y) \leq \mu(X)$ hold. Moreover, we write $\mu(X) < \mu(Y)$ if $\mu(X) \leq \mu(Y)$ and $\mu(X) \neq \mu(Y)$ hold.

Proof. The relation $\mu(X) = \mu(Y)$ defines an equivalence relation on $\text{ind } \mathcal{A}$ and we denote by $\text{ind } \mathcal{A}/\mu$ the set of equivalence classes. This set is partially ordered via μ . The canonical map $\text{ind } \mathcal{A} \rightarrow \text{ind } \mathcal{A}/\mu$ is a morphism of partially ordered sets satisfying the conditions (P1) – (P3) from Theorem 1.6. Suppose we have another transitive relation, written $\mu'(X) \leq \mu'(Y)$ for X, Y in $\text{ind } \mathcal{A}$, and satisfying (GR1) – (GR3). We obtain a second morphism $\text{ind } \mathcal{A} \rightarrow \text{ind } \mathcal{A}/\mu'$ of partially ordered sets satisfying the conditions (P1) – (P3), and we deduce from Theorem 1.6 that for all X, Y

$$\mu'(X) \leq \mu'(Y) \iff \mu(X) \leq \mu(Y). \quad \square$$

Example. (1) Let $X \in \mathcal{A}$ be *uniserial*, that is, X has a unique composition series. Then $\ell^*(X) = \{1, 2, \dots, \ell(X)\}$.

(2) Let $X \in \mathcal{A}$ be an indecomposable object of length at most three. Then

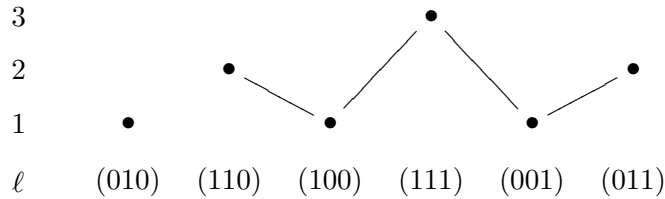
$$\ell^*(X) = \begin{cases} \{1\} & \text{if } \ell(X) = 1, \\ \{1, 2\} & \text{if } \ell(X) = 2, \\ \{1, 2, 3\} & \text{if } \ell(X) = 3 \text{ and } \ell(\text{soc } X) = 1, \\ \{1, 3\} & \text{if } \ell(X) = 3 \text{ and } \ell(\text{soc } X) \neq 1. \end{cases}$$

Here, $\text{soc } X$ denotes the *socle* of X , that is, the sum of all simple subobjects.

(3) Let k be a field and consider the category \mathcal{A} of k -linear representations of the following quiver.

$$1 \longleftarrow 2 \longrightarrow 3$$

An indecomposable representation $V_1 \leftarrow V_2 \rightarrow V_3$ is determined by its dimension vector $(d_1 d_2 d_3)$, where $d_i = \dim_k V_i$. The following Hasse diagram displays the partial order on $\text{ind } \mathcal{A}$, where the layer indicates the length of each object.



From this diagram one computes the Gabriel-Roiter measure $\ell^*(X)$ of each indecomposable object X and obtains the following ordering:

$$\ell^*(010) = \ell^*(100) = \ell^*(001) = \{1\} < \ell^*(111) = \{1, 3\} < \ell^*(110) = \ell^*(011) = \{1, 2\}$$

3.2. Basic properties. Recall from (1.5) that we have established the following property of the Gabriel-Roiter measure.

(GR4) $\mu(X) \leq \mu(Y)$ or $\mu(Y) \leq \mu(X)$ for X, Y in $\text{ind } \mathcal{A}$.

(GR5) $\{\mu(X) \mid X \in \text{ind } \mathcal{A} \text{ and } \ell(X) \leq n\}$ is finite for all $n \in \mathbb{N}$.

Next we discuss further properties of the Gabriel-Roiter measure which depend on the fact that \mathcal{A} is a length category.

3.3. Gabriel-Roiter filtrations. Let $X, Y \in \text{ind } \mathcal{A}$. We say that X is a *Gabriel-Roiter predecessor* of Y if $X \subset Y$ and $\mu(X) = \max_{Y' \subset Y} \mu(Y')$. Note that each object $Y \in \text{ind } \mathcal{A}$ which is not simple admits a Gabriel-Roiter predecessor, by (GR4) and (GR5). A Gabriel-Roiter predecessor X of Y is usually not unique, but the value $\mu(X)$ is determined by $\mu(Y)$.

A sequence

$$X_1 \subset X_2 \subset \dots \subset X_{n-1} \subset X_n = X$$

in $\text{ind } \mathcal{A}$ is called a *Gabriel-Roiter filtration* of X if X_1 is simple and X_{i-1} is a Gabriel-Roiter predecessor of X_i for all $1 < i \leq n$. Clearly, each X admits such a filtration and the values $\mu(X_i)$ are uniquely determined by X .

Proposition. *Let $X, Y \in \text{ind } \mathcal{A}$.*

(GR6) *$X \in \text{ind } \mathcal{A}$ is simple if and only if $\mu(X) \leq \mu(Y)$ for all $Y \in \text{ind } \mathcal{A}$.*

(GR7) *Suppose that $\mu(X) < \mu(Y)$. Then there are $Y' \subset Y'' \subseteq Y$ in $\text{ind } \mathcal{A}$ such that Y' is a Gabriel-Roiter predecessor of Y'' with $\mu(Y') \leq \mu(X) < \mu(Y'')$ and $\ell(Y') \leq \ell(X)$.*

Proof. For (GR6), one uses that each indecomposable object has a simple subobject. To prove (GR7), fix a Gabriel-Roiter filtration $Y_1 \subset Y_2 \subset \dots \subset Y_n = Y$ of Y . We have $\mu(Y_1) \leq \mu(X)$ because Y_1 is simple. Using (GR4), there exists some i such that $\mu(Y_i) \leq \mu(X) < \mu(Y_{i+1})$. Now put $Y' = Y_i$ and $Y'' = Y_{i+1}$. Comparing the filtration of Y with a Gabriel-Roiter filtration of X (as in the proof of Theorem 1.6), we find that $\ell(Y') \leq \ell(X)$. \square

Example. Let $X \in \mathcal{A}$ be uniserial. Then the composition series is a Gabriel-Roiter filtration of X .

3.4. The main property. The following main property of the Gabriel-Roiter measure is crucial for the whole theory.

Proposition (Gabriel). *Let $X, Y_1, \dots, Y_r \in \text{ind } \mathcal{A}$.*

(GR8) *Suppose that $X \subseteq Y = \bigoplus_{i=1}^r Y_i$. Then $\mu(X) \leq \max \mu(Y_i)$ and X is a direct summand of Y if $\mu(X) = \max \mu(Y_i)$.*

Proof. The proof only uses the properties (GR1) – (GR3) of μ . Fix a monomorphism $\phi: X \rightarrow Y$. We proceed by induction on $n = \ell(X) + \ell(Y)$. If $n = 2$, then ϕ is an isomorphism and the assertion is clear. Now suppose $n > 2$. We can assume that for each i the i th component $\phi_i: X \rightarrow Y_i$ of ϕ is an epimorphism. Otherwise choose for each i a decomposition $Y'_i = \bigoplus_j Y_{ij}$ of the image of ϕ_i into indecomposables. Then we use (GR1) and have $\mu(X) \leq \max \mu(Y_{ij}) \leq \max \mu(Y_i)$ because $\ell(X) + \ell(Y') < n$ and $Y_{ij} \subseteq Y_i$ for all j . Now suppose that each ϕ_i is an epimorphism. Thus $\ell(X) \geq \ell(Y_i)$ for all i . Let $X' \subset X$ be a proper indecomposable subobject. Then $\mu(X') \leq \max \mu(Y_i)$ because $\ell(X') + \ell(Y) < n$, and X' is a direct summand if $\mu(X') = \max \mu(Y_i)$. We can exclude the case that $\mu(X') = \max \mu(Y_i)$ because then X' is a proper direct summand of X , which is impossible. Now we apply (GR3) and obtain $\mu(X) \leq \max \mu(Y_i)$. Finally, suppose that $\mu(X) = \max \mu(Y_i) = \mu(Y_k)$ for some k . We claim that we can choose k such that ϕ_k is an epimorphism. Otherwise, replace all Y_i with $\mu(X) = \mu(Y_i)$ by the image $Y'_i = \bigoplus_j Y_{ij}$ of ϕ_i as before. We obtain $\mu(X) \leq \max \mu(Y_{ij}) < \mu(Y_k)$ since $Y_{kj} \subset Y_k$ for all j , using (GR1) and (GR2). This is a contradiction. Thus ϕ_k is an epimorphism

and in fact an isomorphism because $\ell(X) = \ell(Y_k)$ by (GR2). In particular, X is a direct summand of $\oplus_i Y_i$. This completes the proof. \square

Corollary. *Let $X, Y \in \text{ind } \mathcal{A}$ and suppose that $X \subset Y$ with $\mu(X) = \max_{Y' \subset Y} \mu(Y')$. If $X \subseteq U \subset Y$ in \mathcal{A} , then X is a direct summand of U .*

Proof. Let $U = \oplus_i U_i$ be a decomposition into indecomposables. Now apply (GR8). We obtain $\mu(X) \leq \max \mu(U_i) < \mu(Y)$ and our assumption on $X \subset Y$ implies that X is a direct summand of U . \square

Example. (1) Let $Y \in \text{ind } \mathcal{A}$ and suppose that $\mu(X) \leq \mu(Y)$ for all $X \in \text{ind } \mathcal{A}$. Then Y is an injective object, because every monomorphism $Y \rightarrow Z$ splits by (GR8).

(2) Suppose that \mathcal{A} has a cogenerator Q , that is, each object in \mathcal{A} admits a monomorphism into a direct sum of copies of Q . Let $Q = \oplus_i Q_i$ be a decomposition into indecomposable objects. Then $\mu(X) \leq \max \mu(Q_i)$ for all $X \in \text{ind } \mathcal{A}$.

The Gabriel-Roiter measure $\ell^*: \text{ind } \mathcal{A} \rightarrow \text{Ch}(\mathbb{N})$ for \mathcal{A} can be extended to a measure defined for all objects in \mathcal{A} , not only the indecomposable ones. Let $X = \oplus_i X_i$ be an object written as a direct sum of indecomposable objects. Then we define

$$\ell^*(X) = \max \ell^*(X_i).$$

Corollary. *The relation*

$$\mu(X) \leq \mu(Y) \quad :\Longleftrightarrow \quad \ell^*(X) \leq \ell^*(Y) \quad \text{for } X, Y \in \mathcal{A}$$

induces a measure for the set of isomorphism classes of \mathcal{A} .

Proof. We need to verify (M1) – (M3) from Definition 1.1. The first two conditions are automatic and the third is an immediate consequence of (GR8). \square

3.5. Gabriel-Roiter inclusions. Let $X, Y \in \text{ind } \mathcal{A}$. An inclusion $X \subseteq Y$ is called *Gabriel-Roiter inclusion* if $\mu(X) = \max_{Y' \subset Y} \mu(Y')$. Thus we have a Gabriel-Roiter inclusion $X \subseteq Y$ if and only if X is a Gabriel-Roiter predecessor of Y .

Proposition (Ringel). *Let $X, Y \in \text{ind } \mathcal{A}$ and suppose that $X \subset Y$ is a Gabriel-Roiter inclusion. Then Y/X is an indecomposable object.*

Proof. Let $Z = Y/X$ and assume that $Z = Z' \oplus Z''$ with $Z'' \neq 0$. We obtain the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & Z' \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \text{inc} \\
 0 & \longrightarrow & X & \xrightarrow{\text{inc}} & Y & \longrightarrow & Z \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & Z'' & = & Z'' \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

We have $X \subseteq Y' \subset Y$ and therefore the monomorphism $X \rightarrow Y'$ splits by Corollary 3.4. Thus the inclusion $Z' \rightarrow Z$ factors through $Y \rightarrow Z$ via a split monomorphism $Z' \rightarrow Y$. We conclude that $Z' = 0$ since Y is indecomposable. \square

Remark. The argument is borrowed from Auslander and Reiten. They show that the cokernel of an irreducible monomorphism between indecomposable objects is indecomposable.

Corollary. *Let Y be an indecomposable object in \mathcal{A} which is not simple. Then there exists a short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{A} such that X and Z are indecomposable.*

Proof. Take $X \subset Y$ with $\mu(X) = \max_{Y' \subset Y} \mu(Y')$. \square

4. FINITENESS RESULTS

In this section, Ringel's refinement of the first Brauer-Thrall conjecture is presented [4]. More precisely, we prove a structural result about the partial order of the values of the Gabriel-Roiter measure.

4.1. Covariant finiteness. A subcategory \mathcal{C} of \mathcal{A} is called *covariantly finite* if every object $X \in \mathcal{A}$ admits a *left \mathcal{C} -approximation*, that is, a map $X \rightarrow Y$ with $Y \in \mathcal{C}$ such that the induced map $\text{Hom}_{\mathcal{A}}(Y, C) \rightarrow \text{Hom}_{\mathcal{A}}(X, C)$ is surjective for all $C \in \mathcal{C}$. We have also the dual notion: a subcategory \mathcal{C} is *contravariantly finite* if every object in \mathcal{A} admits a *right \mathcal{C} -approximation*.

Lemma. *Let \mathcal{C} be a subcategory of \mathcal{A} which is closed under taking direct sums and subobjects. Then \mathcal{C} is a covariantly finite subcategory of \mathcal{A} .*

Proof. Fix $X \in \mathcal{A}$. Let $X' \subseteq X$ be minimal among the kernels of all maps $X \rightarrow Y$ with $Y \in \mathcal{C}$. Then the canonical map $X \rightarrow X/X'$ is a left \mathcal{C} -approximation. \square

Remark. The proof shows that the inclusion functor $\mathcal{C} \rightarrow \mathcal{A}$ admits a left adjoint $F: \mathcal{A} \rightarrow \mathcal{C}$ which takes $X \in \mathcal{A}$ to X/X' . Note that the adjunction map $X \rightarrow FX$ is a left \mathcal{C} -approximation.

Let M be any set of values $\mu(X)$. Then we define the subcategory

$$\mathcal{A}(M) := \{X \in \mathcal{A} \mid \mu(X) \in M\}.$$

Proposition (Ringel). *Let M be a set of values $\mu(X)$ which is closed under predecessors, that is, $\mu(X_1) \leq \mu(X_2)$ and $\mu(X_2) \in M$ implies $\mu(X_1) \in M$. Then $\mathcal{A}(M)$ is a covariantly finite subcategory of \mathcal{A} .*

Proof. The subcategory $\mathcal{A}(M)$ is closed under taking subobjects by (GR8). \square

4.2. Almost split morphisms. A map $\phi: X \rightarrow Y$ in \mathcal{A} is called *left almost split* if ϕ is not a split monomorphism and every map $X \rightarrow Y'$ in \mathcal{A} which is not a split monomorphism factors through ϕ . Dually, a map $\psi: Y \rightarrow Z$ is called *right almost split* if ψ is not a split epimorphism and every map $Y' \rightarrow Z$ which is not a split epimorphism factors through ψ . For example, if $\mathcal{A} = \text{mod } \Lambda$ for some artin algebra Λ , then every indecomposable object $X \in \mathcal{A}$ admits a left almost split map starting at X and a right almost split map ending at X ; see [1, Cor. V.1.17].

4.3. Immediate successors. Let $X \in \text{ind } \mathcal{A}$. An *immediate successor* of $\mu(X)$ is by definition a minimal element in

$$\{\mu(Y) \mid Y \in \text{ind } \mathcal{A} \text{ and } \mu(X) < \mu(Y)\}.$$

Lemma. *Let $X, Y \in \text{ind } \mathcal{A}$ and suppose that X is a Gabriel-Roiter predecessor of Y . If $X \rightarrow \bar{X}$ is a left almost split map in \mathcal{A} , then Y is a factor object of \bar{X} .*

Proof. The monomorphism $X \rightarrow Y$ factors through $X \rightarrow \bar{X}$ via a map $\phi: \bar{X} \rightarrow Y$. Let U be the image of ϕ . Applying Corollary 3.4, we find that $U = Y$. \square

Proposition. *Let $X \in \text{ind } \mathcal{A}$ and suppose there exists $n_X \in \mathbb{N}$ such that each $V \in \text{ind } \mathcal{A}$ with $\mu(V) \leq \mu(X)$ and $\ell(V) \leq \ell(X)$ admits a left almost split map $V \rightarrow \bar{V}$ with $\ell(\bar{V}) \leq n_X$. Then there exists an immediate successor of $\mu(X)$ provided that $\mu(X)$ is not maximal.*

Proof. Let $\mu(X) < \mu(Y)$. We apply (GR7) and find $Y' \subset Y'' \subseteq Y$ in $\text{ind } \mathcal{A}$ such that Y' is a Gabriel-Roiter predecessor of Y'' with $\mu(Y') \leq \mu(X) < \mu(Y'') \leq \mu(Y)$ and $\ell(Y') \leq \ell(X)$. The preceding lemma implies $\ell(Y'') \leq n_X$, and (GR5) implies that the number of values $\mu(Y'')$ is finite. Thus there exists a minimal element among those $\mu(Y'')$. \square

Corollary (Ringel). *Let Λ be an artin algebra and $X \in \text{ind } \Lambda$. Then there exists an immediate successor of $\mu(X)$ provided that $\mu(X)$ is not maximal.*

Proof. Use that there exists $n_\Lambda \in \mathbb{N}$ having the following property: for each indecomposable $V \in \text{mod } \Lambda$, there exists a left almost split map $V \rightarrow \bar{V}$ satisfying $\ell(\bar{V}) \leq n_\Lambda \ell(V)$. In fact, one takes $n_\Lambda = pq$, where p denotes the maximal length of an indecomposable projective Λ -module and q denotes the maximal length of an indecomposable injective Λ -module; see [1, Prop. V.6.6]. \square

4.4. A finiteness criterion. We present a criterion for a subcategory \mathcal{C} of \mathcal{A} such that the number of indecomposable objects in \mathcal{C} is finite. This is based on the following classical lemma.

Lemma (Harada-Sai). *Let $n \in \mathbb{N}$. A composition $X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_{2^n}$ of non-invertible maps between indecomposable objects of length at most n is zero.*

Proof. See [1, Cor. VI.1.3]. \square

Proposition. *Let \mathcal{A} be a length category with left almost split maps and only finitely many isomorphism classes of simple objects. Suppose that \mathcal{C} is a subcategory such that*

- (1) \mathcal{C} is covariantly finite, and
- (2) there exists $n \in \mathbb{N}$ such that $\ell(X) \leq n$ for all indecomposable $X \in \mathcal{C}$.

Then there are only finitely many isomorphism classes of indecomposable objects in \mathcal{C} .

Proof. We claim that we can construct all indecomposable objects $X \in \mathcal{C}$ in at most 2^n steps from the finitely many simple objects in \mathcal{A} as follows. Choose a non-zero map $S \rightarrow X$ from a simple object S and factor this map through the left \mathcal{C} -approximation $S \rightarrow S'$. Take an indecomposable direct summand X_0 of S' such that the component $S \rightarrow X_0 \rightarrow X$ of the composition $S \rightarrow S' \rightarrow X$ is non-zero. Stop if $X_0 \rightarrow X$ is an isomorphism. Otherwise take a left almost split map $X_0 \rightarrow Y_0$ and a left \mathcal{C} -approximation $Y_0 \rightarrow Z_0$. The map $X_0 \rightarrow X$ factors through the composition $X_0 \rightarrow Y_0 \rightarrow Z_0$ and we choose an

indecomposable direct summand X_1 of Z_0 such that the component $X_0 \rightarrow Y_0 \rightarrow X_1 \rightarrow X$ is non-zero. Again, we stop if $X_1 \rightarrow X$ is an isomorphism. Otherwise, we continue as before and obtain in step r a sequence of non-invertible maps

$$X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_r$$

such that the composition is non-zero. The Harada-Sai lemma implies that $r < 2^n$ because $\ell(X_i) \leq n$ for all i by our assumption. Thus X is isomorphic to X_i for some $i < 2^n$, and we obtain X in at most 2^n steps, having in each step only finitely many choices by taking an indecomposable direct summand. We conclude that \mathcal{C} has only a finite number of indecomposable objects. \square

Remark. This classical argument provides a quick proof of the first Brauer-Thrall conjecture; it is due to Auslander and Yamagata.

4.5. The initial segment.

Theorem (Ringel). *Let \mathcal{A} be a length category such that $\text{ind } \mathcal{A}$ is infinite. Suppose also that \mathcal{A} has only finitely many isomorphism classes of simple objects and that every indecomposable object admits a left almost split map. Then there exist infinitely many values $\mu(X_1) < \mu(X_2) < \mu(X_3) < \dots$ of the Gabriel-Roiter measure for \mathcal{A} having the following properties.*

- (1) *If $\mu(X) \neq \mu(X_i)$ for all i , then $\mu(X_i) < \mu(X)$ for all i .*
- (2) *The set $\{X \in \text{ind } \mathcal{A} \mid \mu(X) = \mu(X_i)\}$ is finite for all i .*

Proof. We construct the values $\mu(X_i)$ by induction as follows. Take for X_1 any simple object. Observe that $\mu(X_1)$ is minimal among all $\mu(X)$ by (GR6) and that only finitely many $X \in \text{ind } \mathcal{A}$ satisfy $\mu(X) = \mu(X_1)$ because \mathcal{A} has only finitely many simple objects. Now suppose that $\mu(X_1) < \dots < \mu(X_n)$ have been constructed, satisfying the conditions (1) and (2) for all $1 \leq i \leq n$. We can apply Proposition 4.3 and find an immediate successor $\mu(X_{n+1})$ of $\mu(X_n)$. It remains to show that the set $\{X \in \text{ind } \mathcal{A} \mid \mu(X) = \mu(X_{n+1})\}$ is finite. To this end consider $M = \{\mu(X_1), \dots, \mu(X_{n+1})\}$. We know from Proposition 4.1 that $\mathcal{A}(M)$ is a covariantly finite subcategory. Clearly, $\ell(X)$ is bounded by $\max\{\ell(X_1), \dots, \ell(X_{n+1})\}$ for all indecomposable $X \in \mathcal{A}(M)$ by (GR2). We conclude from Proposition 4.4 that the number of indecomposables in $\mathcal{A}(M)$ is finite. Thus $\{X \in \text{ind } \mathcal{A} \mid \mu(X) = \mu(X_{n+1})\}$ is finite and the proof is complete. \square

Corollary (Brauer-Thrall I). *Let \mathcal{A} be a length category satisfying the above conditions. Then for every $n \in \mathbb{N}$ there exists an indecomposable object $X \in \mathcal{A}$ with $\ell(X) > n$.*

Proof. Use that for fixed $n \in \mathbb{N}$, there are only finitely many values $\mu(X)$ with $\ell(X) \leq n$, by (GR5). \square

4.6. The terminal segment.

Theorem (Ringel). *Let \mathcal{A} be a length category such that $\text{ind } \mathcal{A}$ is infinite. Suppose also that \mathcal{A} has a cogenerator (i.e. an object Q such that each object in \mathcal{A} admits a monomorphism into a direct sum of copies of Q) and that every indecomposable object admits a right almost split map. Then there exist infinitely many values $\mu(X^1) > \mu(X^2) > \mu(X^3) > \dots$ of the Gabriel-Roiter measure for \mathcal{A} having the following properties.*

- (1) *If $\mu(X) \neq \mu(X^i)$ for all i , then $\mu(X^i) > \mu(X)$ for all i .*
- (2) *The set $\{X \in \text{ind } \mathcal{A} \mid \mu(X) = \mu(X^i)\}$ is finite for all i .*

The proof is based on the following lemma.

Lemma (Auslander-Smalø). *Let \mathcal{A} be a length category and let $X \in \mathcal{A}$. Denote by \mathcal{A}_X the subcategory formed by all objects in \mathcal{A} having no indecomposable direct summand which is isomorphic to a direct summand of X . If every indecomposable direct summand of X admits a right almost split map, then \mathcal{A}_X is contravariantly finite.*

Proof. Let $X = \bigoplus_{i_0=1}^r X_{i_0}$ be a decomposition into indecomposables. It is sufficient to construct a right \mathcal{A}_X -approximation for each indecomposable object $Z \in \mathcal{A}$. We take the identity map if $Z \in \mathcal{A}_X$. Otherwise, Z is isomorphic to X_{i_0} for some i_0 and we proceed as follows. Let $\phi_{i_0} : \bar{X}_{i_0} \rightarrow X_{i_0}$ be a right almost split map and choose a decomposition

$$\bar{X}_{i_0} = Y_{i_0} \oplus (\oplus_{i_1} X_{i_0 i_1})$$

such that $Y_{i_0} \in \mathcal{A}_X$ and $i_0 i_1 \in \{1, \dots, r\}$ for all i_1 . Note that each map $V \rightarrow X_{i_0}$ with $V \in \mathcal{A}_X$ factors through ϕ_{i_0} . Also, each component $X_{i_0 i_1} \rightarrow X_{i_0}$ of ϕ_{i_0} is non-invertible. Now compose ϕ_{i_0} with $\text{id}_{Y_{i_0}} \oplus (\oplus_{i_1} \phi_{i_0 i_1})$ to obtain a map

$$Y_{i_0} \oplus (\oplus_{i_1} (Y_{i_0 i_1} \oplus (\oplus_{i_2} X_{i_0 i_1 i_2}))) \rightarrow Y_{i_0} \oplus (\oplus_{i_1} X_{i_0 i_1}) \rightarrow X_{i_0}.$$

Again, each map $V \rightarrow X_{i_0}$ with $V \in \mathcal{A}_X$ factors through this new map, and each component $X_{i_0 i_1 i_2} \rightarrow X_{i_0 i_1}$ is non-invertible. We continue this procedure, compose this map with

$$\text{id}_{Y_{i_0}} \oplus (\oplus_{i_1} (\text{id}_{Y_{i_0 i_1}} \oplus (\oplus_{i_2} \phi_{i_0 i_1 i_2}))),$$

and so on. Now let $n = 2^m$ where $m = \max\{\ell(X_1), \dots, \ell(X_r)\}$. Then the Harada-Sai lemma implies that any composition

$$X_{i_0 i_1 \dots i_n} \rightarrow X_{i_0 i_1 \dots i_{n-1}} \rightarrow \dots \rightarrow X_{i_0 i_1} \rightarrow X_{i_0}$$

is zero. Thus the induced map

$$\bigoplus_{j=0}^n (\oplus_{i_1, i_2, \dots, i_j} Y_{i_0 i_1 \dots i_j}) \longrightarrow X_{i_0}$$

is a right \mathcal{A}_X -approximation of X_{i_0} . □

Proof of the theorem. We construct the values $\mu(X^i)$ by induction as follows. Let $n \geq 0$ and suppose that $\mu(X^1) > \dots > \mu(X^n)$ have been constructed, satisfying the conditions (1) and (2) for all $1 \leq i \leq n$. Denote by P the direct sum of all $X \in \text{ind } \mathcal{A}$ with $\mu(X) \geq \mu(X^n)$, and let $P = 0$ if $n = 0$. Choose a right \mathcal{A}_P -approximation $P' \rightarrow Q$ and take for X^{n+1} any indecomposable direct summand X of P' such that $\mu(X)$ is maximal. Observe that every indecomposable object $X \in \mathcal{A}_P$ is cogenerated by Q and therefore by P' . Thus (GR8) implies that $\mu(X)$ is bounded by $\mu(X^{n+1})$. Moreover, if $\mu(X) = \mu(X^{n+1})$, then X is isomorphic to a direct summand of P' . Thus $\{X \in \text{ind } \mathcal{A} \mid \mu(X) = \mu(X^{n+1})\}$ is finite and the proof is complete. □

Let Λ be an artin algebra of infinite representation type. Then $\mathcal{A} = \text{mod } \Lambda$ satisfies the assumptions of Theorems 4.5 and 4.6. Let us summarize the structure of the partial order on the values of the Gabriel-Roiter measure as follows. We have

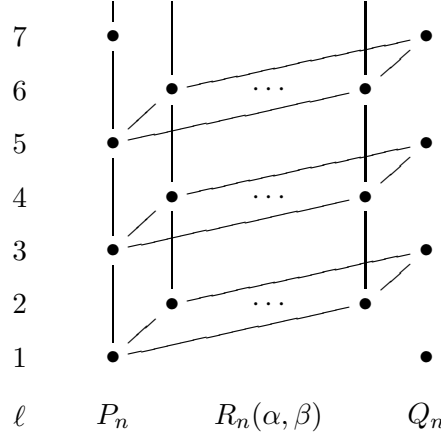
$$\text{ind } \mathcal{A} / \mu := \{\mu(X) \mid X \in \text{ind } \mathcal{A}\} = S_{\text{init}} \sqcup S_{\text{cent}} \sqcup S_{\text{term}} \cong \mathbb{N} \sqcup S_{\text{cent}} \sqcup \mathbb{N}^{\text{op}},$$

where the notation $S = S_1 \sqcup S_2$ for a poset S means $S = S_1 \cup S_2$ and $x_1 < x_2$ for all $x_1 \in S_1, x_2 \in S_2$.

4.7. The Kronecker algebra. Let $\Lambda = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$ be the Kronecker algebra over an algebraically closed field k . We consider the abelian length category which is formed by all finite dimensional Λ -modules. A complete list of indecomposable objects is given by the preprojectives P_n , the regulars $R_n(\alpha, \beta)$, and the preinjectives Q_n ; see [1, Thm. VIII.7.5]. More precisely,

$$\text{ind } \Lambda = \{P_n \mid n \in \mathbb{N}\} \cup \{R_n(\alpha, \beta) \mid n \in \mathbb{N}, (\alpha, \beta) \in \mathbb{P}_k^1\} \cup \{Q_n \mid n \in \mathbb{N}\},$$

and we obtain the following Hasse diagram.



The set of indecomposables is ordered via the Gabriel-Roiter measure as follows:

$$\begin{aligned} \mu(Q_1) = \mu(P_1) < \mu(P_2) < \mu(P_3) < \dots < \mu(R_1) < \mu(R_2) < \mu(R_3) < \dots \\ & \dots < \mu(Q_4) < \mu(Q_3) < \mu(Q_2) \end{aligned}$$

5. THE GABRIEL-ROITER MEASURE FOR DERIVED CATEGORIES

Let \mathcal{A} be an abelian length category. We propose a definition of the Gabriel-Roiter measure for the bounded derived category $\mathbf{D}^b(\mathcal{A})$. The derived Gabriel-Roiter measure extends the Gabriel-Roiter measure for the underlying abelian category \mathcal{A} .

5.1. The definition. The bounded derived category $\mathbf{D}^b(\mathcal{A})$ of \mathcal{A} is by definition the full subcategory of the derived category $\mathbf{D}(\mathcal{A})$ which is formed by all complexes X such that $H^n X = 0$ for almost all n . Note that each object of $\mathbf{D}^b(\mathcal{A})$ admits a finite direct sum decomposition into indecomposable objects having local endomorphism rings. Moreover, such a decomposition is unique up to an isomorphism. We denote by $\text{ind } \mathbf{D}^b(\mathcal{A})$ the set of isomorphism classes of indecomposable objects of $\mathbf{D}^b(\mathcal{A})$.

We consider the functor

$$\mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{A}, \quad X \mapsto H^* X = \bigoplus_{n \in \mathbb{Z}} H^n X,$$

and the isomorphism classes of objects of $\mathbf{D}^b(\mathcal{A})$ are partially ordered via

$$X \leq Y \quad :\Longleftrightarrow \quad \begin{cases} \text{there exists a map } X \rightarrow Y \text{ inducing} \\ \text{a monomorphism } H^* X \rightarrow H^* Y. \end{cases}$$

We have the length function

$$\ell_{H^*}: \text{ind } \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathbb{N}, \quad X \mapsto \ell(H^* X)$$

and the induced chain length function $\ell_{H^*}^*: \text{ind } \mathbf{D}^b(\mathcal{A}) \rightarrow \text{Ch}(\mathbb{N})$ is by definition the *Gabriel-Roiter measure* for $\mathbf{D}^b(\mathcal{A})$.

5.2. Derived versus abelian Gabriel-Roiter measure.

Proposition. *The Gabriel-Roiter measure for $\mathbf{D}^b(\mathcal{A})$ extends the Gabriel-Roiter measure for \mathcal{A} . More precisely, the canonical functor $\mathcal{A} \rightarrow \mathbf{D}^b(\mathcal{A})$ sending an object of \mathcal{A} to the corresponding complex concentrated in degree zero induces an inclusion $\text{ind } \mathcal{A} \rightarrow \text{ind } \mathbf{D}^b(\mathcal{A})$ of partially ordered sets, which makes the following diagram commutative.*

$$\begin{array}{ccc} \text{ind } \mathcal{A} & \xrightarrow{\text{inc}} & \text{ind } \mathbf{D}^b(\mathcal{A}) \\ & \searrow \ell^* & \swarrow \ell_{H^*}^* \\ & \text{Ch}(\mathbb{N}) & \end{array}$$

Proof. Use the fact that the diagram

$$\begin{array}{ccc} \text{ind } \mathcal{A} & \xrightarrow{\text{inc}} & \text{ind } \mathbf{D}^b(\mathcal{A}) \\ & \searrow \ell & \swarrow \ell_{H^*} \\ & \mathbb{N} & \end{array}$$

is commutative and that $\text{ind } \mathcal{A}$ is closed under predecessors in $\text{ind } \mathbf{D}^b(\mathcal{A})$. \square

5.3. An alternative definition. For an alternative definition of the Gabriel-Roiter measure for $\mathbf{D}^b(\mathcal{A})$, consider the lexicographic order on

$$\prod_{\mathbb{Z}} \mathbb{N}_0 := \{(x_n) \in \prod_{\mathbb{Z}} \mathbb{N}_0 \mid x_n = 0 \text{ for almost all } n\}, \text{ with}$$

$$(x_n) \leq (y_n) \iff \begin{cases} x_i = y_i \text{ for all } i \in \mathbb{Z}, \text{ or} \\ x_i \leq y_i \text{ for } i = \min\{n \in \mathbb{Z} \mid x_n \neq y_n\}. \end{cases}$$

Take instead of ℓ_{H^*} the length function

$$\lambda: \text{ind } \mathbf{D}^b(\mathcal{A}) \longrightarrow \prod_{\mathbb{Z}} \mathbb{N}_0, \quad X \mapsto (\ell(H^n X)),$$

and instead of $\ell_{H^*}^*$ the induced chain length function

$$\lambda^*: \text{ind } \mathbf{D}^b(\mathcal{A}) \longrightarrow \text{Ch}\left(\prod_{\mathbb{Z}} \mathbb{N}_0\right).$$

We illustrate the difference between both definitions by taking a hereditary length category \mathcal{A} . Recall that \mathcal{A} is *hereditary* if $\text{Ext}_{\mathcal{A}}^2(-, -) = 0$. Then each indecomposable object of $\mathbf{D}^b(\mathcal{A})$ is isomorphic to a complex concentrated in a single degree. Identifying objects having the same Gabriel-Roiter measure, we obtain

$$\text{ind } \mathbf{D}^b(\mathcal{A}) / \ell_{H^*}^* = \text{ind } \mathcal{A} / \ell^*,$$

whereas

$$\text{ind } \mathbf{D}^b(\mathcal{A}) / \lambda^* = \dots \sqcup \text{ind } \mathcal{A} / \ell^* \sqcup \text{ind } \mathcal{A} / \ell^* \sqcup \text{ind } \mathcal{A} / \ell^* \sqcup \dots$$

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